Derivation of the wave function collapse in the context of Nelson's stochastic mechanics

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Abstract

The von Neumann collapse of the quantum mechanical wavefunction after a position measurement is derived by a purely probabilistic mechanism in the context of Nelson's stochastic mechanics.

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I Introduction

Nelson's stochastic mechanics [1, 2, 3, 4, 5] is a quantization procedure for classical dynamical systems based on stochastic processes of the diffusion type. This theory leads to predictions that agree with those of standard quantum mechanics and are confirmed by experiment. The fundamental assumption is that interaction with a background field causes the system to undergo a diffusion process with diffusion coefficient $\frac{\hbar}{m}$. A fascinating hypothesis concerning the origin of the underlying Brownian motion has been recently advanced by Francesco Calogero in [6]. Namely, that this "tremor" may be caused by the interaction of every particle with the gravitational force due to all other particles of the Universe. Following this idea, he obtains a formula for Planck's action constant h. The latter yields the correct order of magnitude for h when current cosmological data are employed.

It is hardly surprising that the most controversial issue in stochastic mechanics is the measurement problem. Indeed, in [7], Francesco Guerra writes: "Therefore, we see that the basic problem in the interpretation of stochastic mechanics is related to the basic problem in the interpretation of quantum mechanics: To evaluate the effects of the measurement and explain the mechanism of the wave packet reduction".

The purpose of this paper is to show that, in the frame of Nelson's stochastic mechanics, *the wave function reduction after a position measurement may be obtained through a purely probabilistic mechanism*, namely a stochastic variational principle. The latter has the appealing interpretation of changing the pair of forward and backward drifts of the reference process as little as possible given the result of the measurement. This variational principle is quite similar to the one that yields the new stochastic model after measurement for nonequilibrium thermodynamical systems, see Section 5, the only difference being that, in view of the time-reversibility of stochastic mechanics, a time-symmetric kinematics has to be employed. As we have shown elsewhere [8, 9, 10], this kinematics also permits to develop in a natural way a Lagrangian and a Hamiltonian formalism in stochastic mechanics. In particular, it permits to define a *momentum process* having the same first and second moment of the corresponding quantum momentum operator. It is then possible to derive a stochastic counterpart of Hamilton's canonical equations, and to obtain a simple probabilistic interpretation of the uncertainty principle [9] along the lines of [1, 11, 12, 13].

II Kinematics of finite-energy diffusions

In this section, we review some essential concepts and results of the kinematics of diffusion processes. We refer the reader to [2]- [15], [3, 16, 17, 18] for a thorough account. Let $(\Omega, \mathcal{E}, \mathbf{P})$ be a probability space, and let I_n denote the $n \times n$ identity matrix. A stochastic process $\{\xi(t); t_0 \leq t \leq t_1\}$ mapping $[t_0, t_1]$ into $L^2_n(\Omega, \mathcal{E}, \mathbf{P})$ is called a *finite-energy diffusion* with constant diffusion coefficient $I_n\sigma^2$ if the increments admit the representation

$$\xi(t) - \xi(s) = \int_s^t \beta(\tau)d\tau + \sigma[w_+(t) - w_+(s)], \quad t_0 \leq s < t \leq t_1, \tag{II.1}$$

where the *forward drift* $\beta(t)$ is at each time t a measurable function of the past $\{\xi(\tau); 0 \leq \tau \leq t\}$, and $w_+(\cdot)$ is a standard, n-dimensional *Wiener process* with the property that $w_+(t) - w_+(s)$ is independent of $\{\xi(\tau); 0 \leq \tau \leq s\}$. Moreover, β must satisfy the finite-energy condition

$$E\left\{\int_{t_0}^{t_1} \beta(t) \cdot \beta(t)dt\right\} < \infty. \tag{II.2}$$

In [16], Föllmer has shown that a finite-energy diffusion also admits a reverse-time differential. Namely, there exists a measurable function $\gamma(t)$ of the future $\{\xi(\tau); t \leq \tau \leq t_1\}$ called *backward drift*, and another Wiener process w_- such that

$$\xi(t) - \xi(s) = \int_s^t \gamma(\tau) d\tau + \sigma[w_-(t) - w_-(s)], \quad t_0 \leq s < t \leq t_1. \tag{II.3}$$

Moreover, γ satisfies

$$E\left\{\int_{t_0}^{t_1} \gamma(t) \cdot \gamma(t) dt\right\} < \infty, \tag{II.4}$$

and $w_-(t) - w_-(s)$ is independent of $\{\xi(\tau); t \leq \tau \leq t_1\}$. Let us agree that dt always indicates a strictly positive variable. For any function f defined on $[t_0, t_1]$, let

$$d_+ f(t) := f(t + dt) - f(t)$$

be the *forward increment* at time t, and

$$d_- f(t) = f(t) - f(t - dt)$$

be the *backward increment* at time t. For a finite-energy diffusion, Föllmer has also shown in [16] that the forward and backward drifts may be obtained as Nelson's conditional derivatives, namely

$$\beta(t) = \lim_{dt \searrow 0} E\left\{\frac{d_+\xi(t)}{dt} | \xi(\tau), t_0 \leq \tau \leq t\right\}, \tag{II.5}$$

and

$$\gamma(t) = \lim_{dt \searrow 0} E\left\{\frac{d_-\xi(t)}{dt} | \xi(\tau), t \leq \tau \leq t_1\right\}, \tag{II.6}$$

the limits being taken in $L_n^2(\Omega, \mathcal{B}, P)$. It was finally shown in [16] that the one-time probability density $\rho(\cdot, t)$ of $\xi(t)$ (which exists for every $t > t_0$) is absolutely continuous on $\mathbb{R}^n$ and the following relation holds a.s. $\forall t > 0$

$$E\{\beta(t) - \gamma(t) | \xi(t)\} = \sigma^2 \nabla \log \rho(\xi(t), t). \tag{II.7}$$

Let ξ be a finite-energy diffusion satisfying (II.1) and (II.3). Let $f : \mathbb{R}^n \times [t_0, t_1] \to \mathbb{R}$ be twice continuously differentiable with respect to the spatial variable and once with respect to time. Then, we have the following change of variables formulas:

$$f(\xi(t),t) - f(\xi(s),s) = \int_s^t \left(\frac{\partial}{\partial\tau} + \beta(\tau)\cdot\nabla + \frac{\sigma^2}{2}\Delta\right) f(\xi(\tau),\tau)d\tau \tag{II.8}$$

$$+ \int_s^t \sigma\nabla f(\xi(\tau),\tau)\cdot d_+w_+(\tau), \tag{II.9}$$

$$f(\xi(t),t) - f(\xi(s),s) = \int_s^t \left(\frac{\partial}{\partial\tau} + \gamma(\tau)\cdot\nabla - \frac{\sigma^2}{2}\Delta\right) f(\xi(\tau),\tau)d\tau \tag{II.10}$$

$$+ \int_s^t \sigma\nabla f(\xi(\tau),\tau)\cdot d_-w_-(\tau). \tag{II.11}$$

The stochastic integrals appearing in (II.9) and (II.11) are a (forward) Ito integral and a backward Ito integral, respectively, see [15] for the details. Let us introduce the *current drift* $v(t) := (\beta(t)+\gamma(t))/2$ and the *osmotic drift* $u(t) := (\beta(t)-\gamma(t))/2$. Notice that, when σ tends to zero, v tends to $\dot{\xi}$, and u tends to zero. The semi-sum and the semi-difference of (II.9) and (II.11) give two more useful formulas:

$$\begin{aligned} f(\xi(t),t) - f(\xi(s),s) &= \int_s^t \left(\frac{\partial}{\partial\tau} + v(\tau)\cdot\nabla\right) f(\xi(\tau),\tau)d\tau \\ &+\frac{\sigma}{2}\left[\int_s^t \nabla f(\xi(\tau),\tau)\cdot d_+w_+ + \int_s^t \nabla f(\xi(\tau),\tau)\cdot d_-w_-\right], \end{aligned} \tag{II.12}$$

$$\begin{aligned} 0 &= \int_s^t \left(u(\tau)\cdot\nabla + \frac{\sigma^2}{2}\Delta\right) f(\xi(\tau),\tau)d\tau \\ &+\frac{\sigma}{2}\left[\int_s^t \nabla f(\xi(\tau),\tau)\cdot d_+w_+ - \int_s^t \nabla f(\xi(\tau),\tau)\cdot d_-w_-\right]. \end{aligned} \tag{II.13}$$

Specializing (II.12) and (II.13) to $f(x,t) = x$, we get

$$\xi(t) - \xi(s) = \int_s^t v(\tau)d\tau + \frac{\sigma}{2}\left[w_+(t) - w_+(s) + w_-(t) - w_-(s)\right], \tag{II.14}$$

$$0 = \int_s^t u(\tau)d\tau + \frac{\sigma}{2}\left[w_+(t) - w_+(s) - w_-(t) + w_-(s)\right] \tag{II.15}$$

The finite-energy diffusion $\xi(\cdot)$ is called *Markovian* if there exist two measurable functions $b_+(\cdot,\cdot)$ and $b_-(\cdot,\cdot)$ such that $\beta(t) = b_+(\xi(t),t)$ a.s. and $\gamma(t) = b_-(\xi(t),t)$ a.s., for all t in $[t_0,t_1]$. The duality relation (II.7) now reads

$$b_+(\xi(t),t) - b_-(\xi(t),t) = \sigma^2 \nabla \log \rho(\xi(t),t). \tag{II.16}$$

This immediately gives the *osmotic equation*

$$u(x,t) = \frac{\sigma^2}{2} \nabla \log \rho(x,t), \tag{II.17}$$

where $u(x,t) := (b_+(x,t) - b_-(x,t))/2$. The probability density $\rho(\cdot,\cdot)$ of $\xi(t)$ satisfies (at least weakly) the *Fokker-Planck equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (b_+ \rho) = \frac{\sigma^2}{2} \Delta \rho.$$

The latter can also be rewritten, in view of (II.16), as the *equation of continuity* of hydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (v \rho) = 0, \tag{II.18}$$

where $v(x,t) := (b_+(x,t) + b_-(x,t))/2$.

III A time-symmetric kinematics for diffusion processes

We recall here the basic facts from the time-symmetric kinematics developed in [8, 19]. Let us multiply (II.15) by $-i$, and add it to (II.14). We get

$$\begin{aligned} &\xi(t) - \xi(s) = \int_s^t [v(\tau) - iu(\tau)] d\tau \\ &+ \frac{\sigma}{2} \left[(1-i)(w_+(t) - w_+(s)) + (1+i)(w_-(t) - w_-(s)) \right]. \end{aligned} \tag{III.19}$$

We call $v_q(t) := v(t) - iu(t)$ the *quantum drift*, and

$$w_q(t) := \frac{1-i}{2} w_+(t) + \frac{1+i}{2} w_-(t) \tag{III.20}$$

the *quantum noise*. Hence, we can rewrite (III.19) as

$$\xi(t) - \xi(s) = \int_s^t v_q(\tau) d\tau + \sigma[w_q(t) - w_q(s)]. \tag{III.21}$$

At first sight, this decomposition of the *real-valued* increments of ξ into the sum of two *complex* quantities might look somewhat odd. Nevertheless, this representation enjoys several important properties.

1. When σ^2 tends to zero, $v - iu$ tends to $\dot{\xi}$.

2. The quantum drift $v_q(t)$ contains at each time t precisely the same information as the pair $(v(t), u(t))$ (or, equivalently, the pair $(\beta(t), \gamma(t))$.

3. The representation (III.21), differently from (II.1) and (II.3) enjoys an important symmetry with respect to time. Indeed, under time reversal, (III.21) transforms into

$$\xi(t) - \xi(s) = \int_s^t \overline{v_q(\tau)} d\tau + \sigma[\overline{w_q(t) - w_q(s)}], \tag{III.22}$$

   where overbar indicates conjugation, see [9, p.145].

The representation (III.21) has proven to be crucial in order to develop a Lagrangian and Hamiltonian dynamics formalism in the context of Nelson's stochastic mechanics, see [8]-[10]. In particular, to develop the second form of Hamilton's principle, the key tool has been a change of variables formula related to representation (III.21). In order to recall such a formula, we need first to define stochastic integrals with respect to the quantum noise w_q. Let us denote by $d_b f(t) := \frac{1-i}{2} d_+ f(t) + \frac{1+i}{2} d_- f(t)$ the *bilateral increment* of f at time t. Then, from (III.20) and (II.15), we get

$$d_+ w_q(t) = \frac{1+i}{\sigma} u(x(t),t)dt + d_+ w_+ + o(dt),$$

$$d_- w_q(t) = \frac{-1+i}{\sigma} u(x(t),t)dt + d_+ w_- + o(dt).$$

These in turn give immediately

$$d_b w_q(t) := \frac{1-i}{2} d_+ w_+(t) + \frac{1+i}{2} d_- w_-(t) + o(dt). \tag{III.23}$$

Let $f(x,t)$ be a measurable, $\mathbb{C}^n$-valued function such that

$$P\left\{\omega : \int_0^T f(\xi(t),t) \cdot \overline{f(\xi(t),t)} dt < \infty\right\} = 1.$$

In view of (III.23), we define

$$\int_s^t f(\xi(\tau),\tau) \cdot d_b w_q(\tau) := \frac{1-i}{2} \int_s^t f(\xi(\tau),\tau) \cdot d_+ w_+(\tau) + \frac{1+i}{2} \int_s^t f(\xi(\tau),\tau) \cdot d_- w_-(\tau).$$

Thus, integration with respect to the bilateral increments of w_q is defined through a linear combination with complex coefficients of a forward and a backward Ito integral. Let $f(x,t)$ be a complex-valued function with real and imaginary parts of class $C^{2,1}$. Then, multiplying (II.13) by $-i$, and then adding it to (II.12), we get the change of variables formula

$$\begin{aligned} f(\xi(t),t) - f(\xi(s),s) &= \int_s^t \left(\frac{\partial}{\partial \tau} + v_q(\tau) \cdot \nabla - \frac{i\sigma^2}{2} \Delta\right) f(\xi(\tau),\tau) d\tau \\ &+ \int_s^t \sigma \nabla f(\xi(\tau),\tau) \cdot d_b w_q(\tau). \end{aligned} \tag{III.24}$$

It is important to understand that this formula, and in particular the coefficient of the Laplacian term, follows from basic probabilistic arguments.

IV The quantum Hamilton principle

Stochastic mechanics may be based, since the fundamental paper by Guerra and Morato [20], on stochastic variational principles of hydrodynamic type. Other versions of the variational principle have been proposed in [4, 14], and in [8]. We outline here the quantum Hamilton principle of [8], since it employs the time-symmetric kinematics of Section 3 that we shall need to derive the wavefunction collapse.

Let $\mathcal{X}_{\rho_1}$ denote the family of all finite-energy, $\mathbb{R}^n$-valued diffusions on $[t_0, t_1]$ with diffusion coefficient $I_n\frac{\hbar}{m}$, and having marginal probability density ρ_1 at time t_1. Let $\mathcal{V}$ denote the family of finite-energy, C^n - valued stochastic processes on $[t_0, t_1]$. Let $L(x, v) := \frac{1}{2}mv \cdot v - V(x)$ be defined on $R^n \times C^n$. Also let S_0 be a complex-valued function on R^n. Consider the problem of extremizing on $(x, v_q) \in (\mathcal{X}_{\rho_1} \times \mathcal{V})$

$$E\left\{\int_{t_0}^{t_1} L(x(t), v_q(t))\, dt + S_0(x(t_0))\right\} \qquad \text{(IV.25)}$$

subject to the constraint that

$$x \text{ has quantum drift (velocity) } v_q. \qquad \text{(IV.26)}$$

Notice that the quadratic term in the Lagrangian may be rewritten in terms of the forward and backward drifts as follows

$$\frac{m}{2}v_q(t) \cdot v_q(t) = \frac{m}{2}[(\frac{1-i}{2}\beta(t) + \frac{1+i}{2}\gamma(t)] \cdot [(\frac{1-i}{2}\beta(t) + \frac{1+i}{2}\gamma(t)] =$$
$$\frac{-im}{4}[\beta(t) \cdot \beta(t) + 2i\beta(t) \cdot \gamma(t) - \gamma(t) \cdot \gamma(t)] = \frac{-im}{4}[(\beta(t) + i\gamma(t)) \cdot (\beta(t) + i\gamma(t))] \qquad \text{(IV.27)}$$

In [8, Section VIII], the following result was established.

Theorem IV.1 *Suppose that $S_q(x, t)$ of class $C^{2,1}$ solves on $[t_0, t_1]$ the initial value problem*

$$\frac{\partial S_q}{\partial t} + \frac{1}{2m}\nabla S_q \cdot \nabla S_q + V(x) - \frac{i\hbar}{2m}\Delta S_q = 0, \qquad \text{(IV.28)}$$

$$S_q(x,t_0)=S_0(x), \tag{IV.29}$$

and satisfies the technical condition

$$E\left\{\int_{t_0}^{t_1}\nabla S_q(x(t),t)\cdot\overline{\nabla S_q(x(t),t)}\,dt\right\}<\infty,\quad \forall x\in\mathcal{X}_{\rho_1}. \tag{IV.30}$$

Then, any $x\in\mathcal{X}_{\rho_1}$ having quantum drift $\frac{1}{m}\nabla S(x(t),t)$ solves the extremization problem.

A crucial role in the proof is played by the change of variables formula (III.24) that here reads

$$\begin{aligned} f(\xi(t),t)-f(\xi(s),s) &= \int_s^t\left(\frac{\partial}{\partial\tau}+v_q(\tau)\cdot\nabla-\frac{i\hbar}{2m}\Delta\right)f(\xi(\tau),\tau)d\tau \\ &+ \int_s^t\sqrt{\frac{\hbar}{m}}\;\nabla f(\xi(\tau),\tau)\cdot d_bw_q(\tau). \end{aligned} \tag{IV.31}$$

Existence of a solution for the apparently complicated nonlinear, complex Cauchy problem (IV.28)-(IV.29) is dealt with as follows. Let $\{\psi(x,t);t_0\le t\le t_1\}$ be the solution of the *Schrödinger equation*

$$\frac{\partial\psi}{\partial t}=\frac{i\hbar}{2m}\Delta\psi-\frac{i}{\hbar}V(x)\psi, \tag{IV.32}$$

with initial condition $\psi_0(x):=\exp\frac{i}{\hbar}S_0(x)$. If $\psi(x,t)$ never vanishes on $\mathbb{R}^n\times[t_0,t_1]$, and satisfies the condition

$$E\left\{\int_{t_0}^{t_1}\nabla\log\psi(x(t),t)\cdot\overline{\nabla\log\psi(x(t),t)}\,dt\right\}<\infty,\quad \forall x\in\mathcal{X}_{\rho_1}, \tag{IV.33}$$

then $S_q(x,t):=\frac{\hbar}{i}\log\psi(x,t)$ satisfies (IV.28)-(IV.29) and (IV.30). If, moreover, $\psi_0(x)$ has L^2 norm 1, and the terminal density satisfies $\rho_1(x,t)=|\psi(x,t_1)|^2$, then there does exist a Markov diffusion having the required quantum drift, namely the *Nelson process* associated to $\{\psi(x,t);t_0\le t\le t_1\}$, and Born's relation $\rho(x,t)=|\psi(x,t)|^2$ holds, see [8] for the details. The construction of the Nelson process corresponding to $\psi(x,t)$ in the case where $\psi(x,t)$ vanishes requires considerable care. It is discussed in [21], [5, Chapter IV], and references therein.

V Measurement in nonequilibrium thermodynamics

In this section, we discuss measurement for nonequilibrium thermodynamical systems. This serves as an introduction to measurement in stochastic mechanics to be discussed in the following section. Consider an open thermodynamical system whose macroscopic evolution is modeled by an n-dimensional Markov diffusion process $\{x(t); t_0 \leq t\}$ with forward Ito differential

$$d_+x(t) = b_+(x(t))dt + \sigma d_+w_+.$$

Let $\rho(x,t)$ denote the probability density of $x(t)$ satisfying the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (b_+\rho) = \frac{\sigma^2}{2}\Delta\rho. \tag{V.34}$$

The *equilibrium state* is given by the Maxwell-Boltzmann distribution law

$$\bar{\rho}(x) = Cexp[-\frac{H(x)}{kT}],$$

where H is the Hamiltonian function, and we have the relation

$$b_+(x) = -\frac{\sigma^2}{2kT}\nabla H(x),$$

where k is Boltzmann's constant and T is the absolute temperature. Suppose that at time t_1 a measurement is made that yields the new probability density $\tilde{\rho}(x,t_1)$. Let $\mathcal{X}_{\tilde{\rho}(t_1)}$ denote the class of finite-energy diffusions on $[t_1,t_2]$ with diffusion coefficient σ^2 and having marginal $\tilde{\rho}(x,t_1)$ at time t_1. Let us pose the following question: Among all processes in $\mathcal{X}_{\tilde{\rho}(t_1)}$, which one should we use to model the macroscopic evolution of the system from t_1 up to t_2? Everybody agrees that we should employ the stochastic process $\{\tilde{x}(t); t_1 \leq t \leq t_2\}$ that has the same forward drift field $b_+(x)$ of the "reference" process x. This is supported by the observation that the new process must have the same equilibrium distribution of the previous one. Let us show that the new process $\{\tilde{x}(t); t_1 \leq t \leq t_2\}$ may be obtained as solution of a variational problem. Assume that the Kullback-Leibler pseudo-distance between $\tilde{\rho}(t_1)$ and $\rho(t_1)$ is finite, namely

$$H(\tilde{\rho}(t_1), \rho(t_1)) := E\left\{\log\frac{\tilde{\rho}(\tilde{x}(t_1), t_1)}{\rho(\tilde{x}(t_1), t_1)}\right\} = \int_{\mathbb{R}^n}\log\frac{\tilde{\rho}(\tilde{x}, t_1)}{\rho(\tilde{x}, t_1)}\tilde{\rho}(x, t_1)dx < \infty.$$

Let $\mathbb{D}_{\tilde{\rho}(t_1)}$ denote the class of probability measures on $\Omega = C([t_1, t_2])$ that are equivalent to the measure P induced by the reference process $\{x(t); t_1 \leq t \leq t_2\}$. For $Q \in \mathbb{D}_{\tilde{\rho}(t_1)}$, let

$$H(Q, P) = E_Q[\log\frac{dQ}{dP}]$$

denote the *relative entropy* of Q with respect to P. It then follows from Girsanov's theorem that [16, 17]

$$H(Q, P) = H(\tilde{\rho}(t_1), \rho(t_1)) + E_Q\left[\int_{t_1}^{t_2}\frac{1}{2\sigma^2}[b_+(\tilde{x}(t)) - \beta^Q(t)]\cdot[b_+(\tilde{x}(t)) - \beta^Q(t)]dt\right].$$

Since $H(\tilde{\rho}(t_1), \rho(t_1))$ is constant over $\mathbb{D}_{\tilde{\rho}(t_1)}$, it trivially follows that the probability measure $\tilde{Q}$ corresponding to the process $\tilde{x}$ having forward drift b_+ minimizes $H(Q, P)$ over $\mathbb{D}_{\tilde{\rho}(t_1)}$. This problem may be interpreted as a problem of large deviation of the empirical distribution according to Schrödinger's original motivation [22, 17]. We consider now an apparently different variational problem that has the same solution as the previous one. We do so, because it is this second form which, in a suitably modified form, applies to the quantum case. Let $\mathcal{X}_{\tilde{\rho}_2}$ denote the family of finite-energy diffusions on $[t_1, t_2]$ with diffusion coefficient σ^2 and having marginal density $\tilde{\rho}_2$ at time t_2. Consider the problem of minimizing with respect to the pair $(\tilde{x}, \gamma)$ the functional

$$E\left\{\int_{t_1}^{t_2}\frac{1}{2\sigma^2}[b_-(\tilde{x}(t)) - \gamma(t)]\cdot[b_-(\tilde{x}(t)) - \gamma(t)]dt - \log\frac{\tilde{\rho}(\tilde{x}(t_1), t_1)}{\rho(\tilde{x}(t_1), t_1)}\right\}$$

subject to the constraint that γ be the backward drift of $\tilde{x}$ on $[t_1, t_2]$. This problem is a variant of the one first considered and solved in [23, Theorem 2]. The connection between the two variational problems, and their relation to the theory of Schrödinger processes and bridges, has been thoroughly investigated in [24]. In order to solve this problem, rather than reproducing the arguments in [23, 24], we take the opportunity to introduce the variational method based on nonlinear Lagrange functionals, [25]. This method permits to solve also the more complicated quantum case. Suppose that we wish to minimize $J : Y \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}}$ denotes the extended reals, over the nonempty subset S of Y.

Lemma V.1 (Lagrange Lemma) *Let $\Lambda : Y \to \bar{\mathbb{R}}$ and let $y_0 \in S$ minimize $J + \Lambda$ over Y. Assume that $\Lambda(\cdot)$ is* finite *and* constant *over S. Then y_0 minimizes J over S.*

Proof. For any $y \in S$, we have $J(y_0) + \Lambda(y_0) \leq J(y) + \Lambda(y) = J(y) + \Lambda(y_0)$. Hence $J(y_0) \leq J(y)$. □

A functional Λ which is constant and finite on S is called a *Lagrange functional.* Obviously, a similar result holds if the problem is an extremization problem. Let us apply this simple idea to the above problem. Let $\varphi(x,t)$ be a real-valued function of class $C^{2,1}$ defined on $\mathbb{R}^n \times [t_1, t_2]$, and satisfying the technical condition

$$E\left\{\int_{t_1}^{t_2} \nabla\varphi(x(t),t)\cdot\nabla\varphi(x(t),t)\,dt\right\} < \infty, \quad \forall x \in \mathcal{X}_{\tilde{\rho}_2}. \tag{V.35}$$

Corresponding to such a φ, we introduce the functional

$$\begin{aligned}\Lambda^{\varphi}(\tilde{x},\gamma) &:= E\left\{\varphi(\tilde{x}(t_2),t_2) - \varphi(\tilde{x}(t_1),t_1)\right.\\ &\left.+ \int_{t_1}^{t_2}\left[-\frac{\partial\varphi}{\partial t}(\tilde{x}(t),t) - \gamma(t)\cdot\nabla\varphi(\tilde{x}(t),t) + \frac{\sigma^2}{2}\Delta\varphi(\tilde{x}(t),t)\right]dt\right\}.\end{aligned}$$

In view of (II.11) and (V.35), we have that $\Lambda^{\varphi}(\tilde{x},\gamma) = 0$ whenever the pair $(\tilde{x},\gamma)$ satisfies the constraint since the stochastic integral has zero expectation. Thus, it is a *Lagrange functional* for the problem. Consider next the *unconstrained* minimization of the functional $J + \Lambda^{\varphi}$. For a fixed $\tilde{x} \in \mathcal{X}_{\tilde{\rho}_2}$, and a fixed time $t \in [t_1, t_2]$, we consider the *pointwise* minimization of the integrand of $J+\Lambda^{\varphi}$ with respect to γ

$$\mathrm{minimize}_{\gamma\in R^n}\{\frac{1}{2\sigma^2}(b_-(\tilde{x}(t),t)-\gamma)\cdot(b_-(\tilde{x}(t),t)-\gamma)-\gamma\cdot\nabla\varphi(\tilde{x}(t),t)\}$$

We get

$$\gamma^o(\tilde{x})(t)=b_-(\tilde{x}(t),t)+\sigma^2\nabla\varphi(\tilde{x}(t),t). \tag{V.36}$$

Substituting back expression (V.36) into $J+\Lambda^{\varphi}$, we get the following functional of $\tilde{x}$

$$\begin{aligned}(J+\Lambda^{\varphi})(\tilde{x},\gamma^o(\tilde{x})) := E\Bigg\{&\varphi(\tilde{x}(t_2),t_2)-\varphi(\tilde{x}(t_1),t_1)-\log\frac{\tilde{\rho}(\tilde{x}(t_1),t_1)}{\rho(\tilde{x}(t_1),t_1)}+\\ &\int_{t_1}^{t_2}\left[-\frac{\sigma^2}{2}\nabla\varphi(\tilde{x}(t),t)\cdot\nabla\varphi(\tilde{x}(t),t)-\frac{\partial\varphi}{\partial t}(\tilde{x}(t),t)\right.\\ &\left.-b_-(\tilde{x}(t),t)\cdot\nabla\varphi(\tilde{x}(t),t)+\frac{\sigma^2}{2}\Delta\varphi(\tilde{x}(t),t)\right]dt\Bigg\}.\end{aligned} \tag{V.37}$$

Next, we seek to find a function φ such that the functional $(J+\Lambda^{\varphi})(\tilde{x},\gamma^o(\tilde{x}))$ becomes constant over $\mathcal{X}_{\tilde{\rho}_2}$. Suppose φ solves on $[t_1,t_2]$ the initial value problem

$$\frac{\partial\varphi}{\partial t}+b_-(x,t)\cdot\nabla\varphi(x,t)-\frac{\sigma^2}{2}\Delta\varphi(x,t)=-\frac{\sigma^2}{2}\nabla\varphi(x,t)\cdot\nabla\varphi(x,t), \tag{V.38}$$

$$\varphi(x,t_1)=-\log\frac{\tilde{\rho}(x,t_1)}{\rho(x,t_1)}. \tag{V.39}$$

Then $(J+\Lambda^{\varphi})(\tilde{x},\gamma^o(x))=E\{\varphi(\tilde{x}(t_2),t_2)\}$ is constant over $\mathcal{X}_{\tilde{\rho}_2}$ since such processes have the same marginal density at time t_2. Hence, any $x\in\mathcal{X}_{\tilde{\rho}_2}$ solves the unconstrained minimization of $J+\Lambda^{\varphi}$. To solve the original constrained problem, we need to find $\tilde{x}\in\mathcal{X}_{\tilde{\rho}_2}$ that has backward drift given by (V.36). In order to do that, we first proceed to find the solution of (V.38)-(V.39). Define $\tilde{\rho}(x,t):=\exp[-\varphi(x,t)]\rho(x,t)$. Then, if φ satisfies (V.38), using the Fokker-Plank equation satisfied by ρ, we get

$$\frac{\partial\tilde{\rho}}{\partial t}=\exp[-\varphi]\left(-\frac{\partial\varphi}{\partial t}\rho+\frac{\partial\rho}{\partial t}\right)=$$

$$\left(b_- \cdot \nabla\varphi - \frac{\sigma^2}{2}\Delta\varphi + \frac{\sigma^2}{2}\nabla\varphi\cdot\nabla\varphi\right)\tilde\rho - \exp[-\varphi]\nabla\cdot(b_+\rho) + \exp[-\varphi]\frac{\sigma^2}{2}\Delta\rho =$$
$$\frac{\sigma^2}{2}\Delta\tilde\rho + b_+\cdot\nabla\varphi\tilde\rho - \exp[-\varphi]\nabla\rho\cdot b_+ - \exp[-\varphi]\rho\nabla\cdot b_+ = -\nabla\cdot(\tilde\rho b_+) + \frac{\sigma^2}{2}\Delta\tilde\rho.$$

We conclude that if $\tilde\rho$ is the solution of the Fokker-Planck equation (V.34) on $[t_1, t_2]$ with initial condition at time t_1 given by $\tilde\rho(x, t_1)$, then $\varphi := -\log\frac{\tilde\rho}{\rho}$ solves the initial value problem (V.38)-(V.39). Thus, we have the following result.

Theorem V.2 *Let $\tilde\rho$ be the solution of the Fokker-Planck equation (V.34) on $[t_1, t_2]$ with initial condition given by $\tilde\rho(x, t_1)$. Then $\varphi := -\log\frac{\tilde\rho}{\rho}$ solves the initial value problem (V.38)-(V.39). Suppose that φ satisfies (V.35), and that $\tilde\rho_2(x) = \tilde\rho(x, t_2)$. Then the stochastic process $\tilde x \in \mathcal{X}_{\tilde\rho_2}$ having backward drift field $\tilde b_-(x,t) = b_-(x,t) - \sigma^2\nabla\log\frac{\tilde\rho}{\rho}(x,t) = b_+ - \sigma^2\nabla\log\tilde\rho(x,t)$ solves the constrained minimization problem.*

In view of (II.16), we see that the solution process has forward drift $b_+(\cdot)$, and therefore coincides with the solution of the previous variational problem. Consider the same problem on the interval $[t_1, t_3]$, where $t_3 > t_2$. If we impose the density $\tilde\rho(x, t_3)$ at the final time, the solution process coincides with the previous solution process up to time t_2. This may be viewed as a form of coherence with respect to the terminal time. It is also important to observe that the new process $\{\tilde x(t); t_1 \leq t \leq t_2\}$ has the same forward drift of the reference process $\{x(t); t_1 \leq t \leq t_2\}$, but a *different backward drift*. Hence, while the forward transition probabilities have been preserved, *the reverse-time transition probabilities have changed*. Thus, we see that it is impossible, even in principle, to estimate the reverse-time transition probabilities by repeated measurement. In [14, 7], Nelson and Guerra regard as a serious drawback of stochastic mechanics the fact that transition probabilities of the Nelson process are not open to experimental verification if we accept that transition probabilities are associated to a definite quantum state. We shall come back to this crucial point in the next section.

VI A stochastic derivation of wave function collapse

In Section 4, we have seen that the Schrödinger equation is obtained through a simple exponential transformation from the Hamilton-Jacobi equation (IV.28) of an appropriate stochastic variational principle. Suppose now that a position measurement of the quantum system is made at time t_1, and we ask: What should be the new stochastic process on $[t_1, t_2]$? First of all, we consider the situation without measurement up to time t_2. In this case, the variational principle of Section 4 would have as solution the Nelson process $\{x(t); t_0 \leq t \leq t_2\}$ extended up to time t_2 with quantum drift $v_q(t) = \frac{\hbar}{im} \nabla \log \psi(x(t), t)$, where $\{\psi(x, t) : t_0 \leq t \leq t_2\}$ is the solution of the Schrödinger equation (IV.32). The Nelson process $\{x(t); t_1 \leq t \leq t_2\}$ will play the role of a "reference process". Suppose that the measurement at time t_1 yields the new probability density $\tilde{\rho}(x, t_1)$. For instance, if we assume that the measurement at time t_1 only gives the information that x lies in a certain subset D of the configuration space of the system, the density $\tilde{\rho}(x, t_1)$ just after the measurement is given, according to Bayes' theorem, by

$$\tilde{\rho}(x, t_1) = \frac{\chi_D(x)\rho(x, t_1)}{\int_D \rho(x', t_1) dx'} \ ,$$

where $\rho(x, t_1)$ is the probability density of the Nelson reference process right before the measurement is made. We need now to find an appropriate variational mechanism that, employing the Nelson reference process and the probability density $\tilde{\rho}(x, t_1)$, produces the new process $\{\tilde{x}(t); t_1 \leq t \leq t_2\}$. It is apparent that the variational mechanism of the previous section is not suitable here. Indeed, as observed before, that mechanism preserves completely the *forward* drift and transition probabilities, but changes, possibly in a dramatic way, the backward drift and transition probabilities. This is not acceptable in stochastic mechanics, were forward and backward drifts and transition probabilities *must always be granted the same status*. In other words, the time-reversibility of the theory must be reflected also by the theory of measurement. On the other hand, preserving both drifts, or equivalently both transition probabilities, amounts to preserving the process $\{x(t); t_0 \leq t \leq t_2\}$, which is impossible since the probability density at time t_1 has changed. Thus, we need to find a variational mechanism that *changes both drifts as little as possible, given the new density at time* t_1. In should be apparent that, at this point, the time-symmetric kinematics of Section 3 is called for. Given that kinematics, and by analogy with the variational principle of the previous section, we are then led to the following formulation.

In the notation of Section 4, we consider the problem of extremizing on $(\tilde{x}, \tilde{v}_q) \in (\mathcal{X}_{\tilde{\rho}_2} \times \mathcal{V})$ the functional

$$J(\tilde{x}, \tilde{v}_q) := E\left\{\int_{t_1}^{t_2} \frac{mi}{2\hbar}(v_q(\tilde{x}(t),t) - \tilde{v}_q(t)) \cdot (v_q(\tilde{x}(t),t) - \tilde{v}_q(t))\, dt + \frac{1}{2}\log\frac{\tilde{\rho}(\tilde{x}(t_1),t_1)}{\rho(\tilde{x}(t_1),t_1)}\right\} \tag{VI.40}$$

subject to the constraint that

$$\tilde{x} \text{ has quantum drift (velocity) } \tilde{v}_q. \tag{VI.41}$$

Here $v_q(x,t) = \frac{\hbar}{im}\nabla \log\psi(x,t)$ is the quantum drift field of the Nelson reference process, and $\mathcal{X}_{\tilde{\rho}_2}$ is the family of all finite-energy, $\mathbb{R}^n$-valued diffusions on $[t_1, t_2]$ with diffusion coefficient $I_n\frac{\hbar}{m}$, and having probability density $\tilde{\rho}_2$ at time t_2. The structure of the functional is quite similar to the one of the previous section. Here, $\frac{\hbar}{mi}$ replaces σ^2 in view of formula

(IV.31). The $\frac{1}{2}$ in the boundary term is justified by the following relation, see (IV.27),

$$\frac{mi}{2\hbar}(v_q(x,t)-\tilde{v}_q(t))\cdot(v_q(x,t)-\tilde{v}_q(t))=$$
$$\frac{m}{4\hbar}\left[(b_+(x,t)-\tilde{b}_+(t))+i(b_-(x,t)-\tilde{b}_-(t))\right]\cdot\left[(b_+(x,t)-\tilde{b}_+(t))+i(b_-(x,t)-\tilde{b}_-(t))\right]$$

which shows that a $\frac{1}{4}$ appears in the right-hand side. To solve this variational problem, we employ the same strategy as in the previous section. Let $\varphi(x,t)$ be a complex-valued function of class $C^{2,1}$ defined on $\mathbb{R}^n\times[t_1,t_2]$, and satisfying the technical condition

$$E\left\{\int_{t_1}^{t_2}\nabla\varphi(x(t),t)\cdot\overline{\nabla\varphi(x(t),t)}\,dt\right\}<\infty,\quad\forall x\in\mathcal{X}_{\rho_2}. \tag{VI.42}$$

Corresponding to such a φ, we introduce the functional

$$\Lambda^{\varphi}(\tilde{x},\tilde{v}_q):=E\left\{\varphi(\tilde{x}(t_2),t_2)-\varphi(\tilde{x}(t_1),t_1)+\right.$$
$$\left.\int_{t_1}^{t_2}\left[-\frac{\partial\varphi}{\partial t}(\tilde{x}(t),t)-\tilde{v}_q(t)\cdot\nabla\varphi(\tilde{x}(t),t)+\frac{i\hbar}{2m}\Delta\varphi(\tilde{x}(t),t)\right]dt\right\}.$$

In view of (III.24), and of property (VI.42), we see that $\Lambda^{\varphi}(\tilde{x},\tilde{v}_q)=0$ whenever the pair $(\tilde{x},\tilde{v}_q)$ satisfies the constraint. Thus, it is a *Lagrange functional* for the problem. Consider next the *unconstrained* extremization of the functional $J+\Lambda^{\varphi}$. For a fixed $\tilde{x}\in\mathcal{X}_{\tilde{\rho}_2}$, and a fixed time $t\in[t_1,t_2]$, we consider the *pointwise* extremization of the integrand of $J+\Lambda^{\varphi}$ with respect to $\tilde{v}_q$

$$\text{extremize}_{\tilde{v}\in C^n}\left\{\frac{mi}{2\hbar}(v_q(\tilde{x}(t),t)-\tilde{v})\cdot(v_q(\tilde{x}(t),t)-\tilde{v})-\tilde{v}\cdot\nabla\varphi(\tilde{x}(t),t)\right\}$$

We get

$$\tilde{v}_q^o(\tilde{x})(t)=v_q(\tilde{x}(t),t)+\frac{\hbar}{mi}\nabla\varphi(\tilde{x}(t),t). \tag{VI.43}$$

Substituting back expression (VI.43) into $J+\Lambda^{\varphi}$, we get the following functional of $\tilde{x}$

$$(J+\Lambda^{\varphi})(\tilde{x},\tilde{v}_q^o(x)):=E\left\{\varphi(\tilde{x}(t_2),t_2)-\varphi(\tilde{x}(t_1),t_1)+\right.$$

$$\int_{t_1}^{t_2}\left[\frac{i\hbar}{2m}\nabla\varphi(\tilde{x}(t),t)\cdot\nabla\varphi(\tilde{x}(t),t)-\frac{\partial\varphi}{\partial t}(\tilde{x}(t),t)\right.$$
$$\left.\left.-v_q(\tilde{x}(t),t)\cdot\nabla\varphi(\tilde{x}(t),t)+\frac{i\hbar}{2m}\Delta\varphi(\tilde{x}(t),t)\right]dt\right\}. \tag{VI.44}$$

We seek next to choose the function φ so that the functional $(J+\Lambda^{\varphi})(\tilde{x},\tilde{v}_q^o(x))$ becomes constant over $\mathcal{X}_{\tilde{\rho}_2}$. Suppose φ solves on $[t_1,t_2]$ the initial value problem

$$\frac{\partial\varphi}{\partial t}+v_q(x,t)\cdot\nabla\varphi(x,t)-\frac{i\hbar}{2m}\Delta\varphi(x,t)=\frac{i\hbar}{2m}\nabla\varphi(x,t)\cdot\nabla\varphi(x,t), \tag{VI.45}$$
$$\varphi(x,t_1)=\frac{1}{2}\log\frac{\tilde{\rho}(x,t_1)}{\rho(x,t_1)}. \tag{VI.46}$$

Then $(J+\Lambda^{\varphi})(\tilde{x},\tilde{v}_q^o(x))=E\{\varphi(\tilde{x}(t_2),t_2)\}$ is constant over $\mathcal{X}_{\tilde{\rho}_2}$ since such processes have have the same marginal density at time t_2. Hence, any $x\in\mathcal{X}_{\tilde{\rho}_2}$ solves the unconstrained extremization of $J+\Lambda^{\varphi}$. To solve the original constrained extremization problem, we need to find $\tilde{x}\in\mathcal{X}_{\tilde{\rho}_2}$ that has quantum drift given by (VI.43). In order to do that, we first proceed to find the solution of (VI.45)-(VI.46). Write $\psi(x,t_1)=\rho(x,t_1)^{\frac{1}{2}}\exp[\frac{i}{\hbar}S(x,t_1)]$, and define $\tilde{\psi}(x,t):=\exp[\varphi(x,t)]\psi(x,t)$. Then, if φ satisfies (VI.45), using the Schrödinger equation (IV.32) satisfied by ψ, we get

$$\frac{\partial\tilde{\psi}}{\partial t}=\exp[\varphi]\left(\frac{\partial\varphi}{\partial t}\psi+\frac{\partial\psi}{\partial t}\right)=$$
$$-\frac{i}{\hbar}V(x)\tilde{\psi}+\frac{i\hbar}{2m}\exp\varphi\left(\Delta\psi+2\nabla\psi\cdot\nabla\varphi+\nabla\varphi\cdot\nabla\varphi\psi+\Delta\varphi\psi\right)=\frac{i\hbar}{2m}\Delta\tilde{\psi}-\frac{i}{\hbar}V(x)\tilde{\psi}.$$

Observing that $\tilde{\psi}(x,t_1)=\tilde{\rho}(x,t_1)^{\frac{1}{2}}\exp[\frac{i}{\hbar}S(x,t_1)]$, we conclude that if $\tilde{\psi}$ is the solution of the Schrödinger equation (IV.32) on $[t_1,t_2]$ with initial condition at time t_1 given by $\tilde{\rho}(x,t_1)^{\frac{1}{2}}\exp[\frac{i}{\hbar}S(x,t_1)]$, then $\varphi:=\log\frac{\tilde{\psi}}{\psi}$ solves the initial value problem (VI.45)-(VI.46). Thus, we get the following result.

Theorem VI.1 *Suppose that $\tilde{\psi}$ is the solution of the Schrödinger equation (IV.32) on $[t_1,t_2]$ with initial condition at time t_1 given by $\tilde{\rho}(x,t_1)^{\frac{1}{2}}\exp[\frac{i}{\hbar}S(x,t_1)]$. Then $\varphi:=\log\frac{\tilde{\psi}}{\psi}$*

solves the initial value problem (VI.45)-(VI.46). Suppose that φ satisfies (VI.42), and that $\tilde{\rho}_2(x) = |\tilde{\psi}(x,t_2)|^2$. Then the stochastic process $\tilde{x} \in \mathcal{X}_{\tilde{\rho}_2}$ having quantum drift $\frac{\hbar}{mi}\nabla \log \tilde{\psi}(\tilde{x}(t),t)$ solves the constrained extremization problem.

Thus, by a purely probabilistic argument, we have shown that the new process after the measurement at time t_1 is associated to another solution $\tilde{\psi}$ of the same Schrödinger equation (IV.32). The association is precisely as before, namely the quantum drift is proportional to the gradient of the logarithm of $\tilde{\psi}$. In other words, the new process is just the Nelson process associated to the solution $\{\tilde{\psi}(x,t); t_1 \leq t \leq t_2\}$. It is important to observe that the new wave function has the same phase at time t_1 as the old one before measurement. This agrees with standard quantum mechanics when it is assumed that immediate repetition of the measurement yields the same result and does not change the wavefunction except for an arbitrary phase factor, see e.g. [26, 27]. Here, however, no further assumption is needed: *The invariance of the phase follows from the variational principle.* This is a crucial point. Indeed, if we assume the invariance of the phase after a position measurement in stochastic mechanics, then the variational principle of Section 4 suffices to produce the new Nelson process (associated to the solution $\{\tilde{\psi}(x,t)\}$ of the Schrödinger equation). Also notice that the solution process possesses the same coherence property with respect to the time interval as the solution process of the previous section.

VII Discussion

In this paper we have shown that, in the frame of Nelson's stochastic mechanics, the wave function reduction does not need to be *postulated*, but may be *derived* from the standard rules of probability (Bayes' theorem) and a stochastic variational principle of transparent significance. It seems to us that this result lends support to the point of view of Blanchard, Golin and Serva in [28], where it was shown that some apparent paradoxes of stochastic mechanics related to repeated measurements could be removed by introducing an appropriate new process after each measurement. The new process, indeed, is the Nelson process associated to the new solution $\tilde{\psi}$ of the Schrödinger equation. A general comparison between standard quantum mechanics and stochastic mechanics is beyond the aims of this paper, and anyway beyond the knowledge and the understanding of the present author. We refer the reader to [4, 14], as well as to a series of recent papers by Francesco Guerra [7, 29], for a thorough and deep analysis on the possibility of regarding Nelson's stochastic mechanics as a complete physical theory.

Nevertheless, it seems legitimate to us to stress that stochastic mechanics, including the elements of a theory of measurement outlined in [28] and here, can simply be based on the hypothesis of universal Brownian motion and on stochastic variational principles. Thus, stochastic mechanics appears as a generalization of classical mechanics whose foundations are completely independent from standard quantum mechanics. Moreover, this theory is now capable of providing a transparent probabilistic derivation of the two most mysterious features of standard quantum mechanics, namely the uncertainty principle and the wave function collapse.

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